The homotopy classes of continuous maps between some non-metrizable manifolds

Mathieu Baillif

keywords: *homotopy, longline, nonmetrizable manifolds*

Subject code classification: 57N99, 14F35

Abstract

We prove that the homotopy classes of continuous maps $\mathsf{R}^n \to \mathsf{R}$, where R is Alexandroff's long ray, are in bijection with the antichains of $\mathcal{P}(\{1,\ldots,n\})$. The proof uses partition properties of continuous maps $\mathsf{R}^n \to \mathsf{R}$. We also provide a description of $[X,\mathsf{R}]$ for some other non-metrizable manifolds.

1 Introduction

This paper is about computation of homotopy classes of maps between some non-metrizable manifolds. The main result is a complete classification of homotopy classes of continuous functions $\mathsf{R}^n \to \mathsf{R}$, where R is Alexandroff's long ray, which are shown to be in bijection with the antichains of $\mathcal{P}(\{1,\ldots,n\})\backslash\{\emptyset\}$ (see Theorems 1–2 below). This generalises a result of D. Gauld [4] who solved the problem when $n = 1$.

Under our opinion, there are (at least) three reasons that motivate this investigation of homotopy in non-metrizable manifolds. Firstly, any manifold is compactly generated, and thus, according to G.W. Whitehead, fits in the natural category of homotopy theory (see e.g. [8]). Secondly, our manifolds provide a class of spaces X for which $\Pi_i(X) = 0$ for each $i \in \omega$ while $[X,X]$ is finite but has at least two elements, so in particular X is not contractible. (Notice that since $\Pi_1(X) = \{0\}$, we do not need to bother about base points, and consider only free homotopy.) The non-contractibility does not come from the "shape" of X but rather from its "wideness". The third reason

(of a more practical nature) is that the proofs are completely elementary, in the sense that we use only very basic facts about countable ordinals. In fact, despite what the title may suggest, the main part of this paper consists in an investigation of partition properties of maps $\mathsf{R}^n \to \mathsf{R}$ (which we find interesting in themselves), which enable us to reduce the purely homotopical questions to the trivial fact that two maps $[0,1]^n \to [0,1]$ are homotopic.

The paper is organised as follows. Section 2 contains the definitions and the statements of the main results. In particular, we define the cofinality class $\mathfrak{C}(f)$ of a map $f : \mathsf{R}^n \to \mathsf{R}$. In Section 3, we devise some properties of "big" open and closed sets in R^n which will often be of use. In Section 4, we show that the cofinality classes of functions $\mathsf{R}^n \to \mathsf{R}$ are in bijection with the antichains of $\mathcal{P}(\{1,\ldots,n\})\backslash\{\emptyset\}$. Then, in Sections 5-6, we prove that f and g are homotopic if and only if $\mathfrak{C}(f) = \mathfrak{C}(g)$. Finally, in Section 7 we investigate some other non-metrizable manifolds.

This paper can be seen as a companion to [2] where D. Cimasoni and I have investigated embeddings $\mathsf{R} \to \mathsf{R}^n$ up to ambiant isotopy.

2 Definitions

We recall that Alexandroff's (closed) long ray R is $\omega_1 \times [0,1[$ endowed with the topology given by the lexicographic order $\leq$. It is well known that R can be made into a 1-dimensional ($\mathcal{C}^\infty$) manifold, is sequentially compact, non-metrizable and non-contractible. In this paper, sequential compacity is the key property and will always be implicitely invoked when we say that some (sub)sequence converge. Two other well known properties of R are given in the following lemmas whose proofs can be found e.g. in [6, Lemma 3.4 (iii)] and [5]:

Lemma 2.1. *Let $f : \mathsf{R} \to \mathsf{R}$ be continuous and bounded. Then, f is eventually constant, i.e. there is $z \in \mathsf{R}$ such that $f(x) = f(z)$ if $x \geq z$.*

Lemma 2.2. *Let $\{E_m\}_{m\in\omega}$ be closed unbounded sets of R. Then, $\cap_{m\in\omega} E_m$ is closed and unbounded.*

(In both lemmas R can be replaced by ω_1.) We will always identify the ordinal $\alpha \in \omega_1$ with $(\alpha, 0) \in \mathsf{R}$, and thus consider ω_1 as a subset of R. We use greek letters for ordinals, and only for them.
Let us fix n, the dimension, and set $N = \{1,\ldots,n\}$. We denote by $\pi_i : \mathsf{R}^n \to \mathsf{R}$ the projection on the i-th coordinate. We will often define sequences in R or R^n, so to avoid confusion we shall use only the index m to denote a member of a sequence while we reserve the

indices i, j, k, ℓ for coordinates. For $x = (x_1, \ldots, x_n) \in \mathsf{R}^n$ we set $|x| = \max_{i=1,\ldots,n} x_i$. For a finite set I, we denote its number of elements by $|I|$. If $I = \{i_1, \ldots, i_k\} \subset N$ and $x \in \mathsf{R}^n$, we write x_I for $(x_{i_1}, \ldots, x_{i_k})$.

Definition 2.3. *Let $I \subset N$ and $c \in \mathsf{R}^{n-|I|}$. The I-diagonale at height c is the set*

$$\Delta_I(c) \overset{\text{def}}{=} \left\{ x = (x_1, \ldots, x_n) \in \mathsf{R}^n \ : \ \begin{array}{c} x_i = x_{i'} \text{ if } i, i' \in I \\ \text{and } x_{N \setminus I} = c \end{array} \right\}. \tag{1}$$

We abbreviate $\Delta_I(0)$ by Δ_I.

Notice that $\Delta_I(c)$ is homeomorphic to R.

Lemma 2.4. *Let $I \subset N$ and $c, c' \in \mathsf{R}^{n-|I|}$. Then, $f\big|_{\Delta_I(c)}$ is unbounded (resp. bounded) if and only if $f\big|_{\Delta_I(c')}$ is unbounded (resp. bounded).*

Proof. Let $\gamma : [0,1] \to \mathsf{R}^{n-|I|}$ be continuous with $\gamma(0) = c$, $\gamma(1) = c'$. Then, γ provides an homotopy between $f\big|_{\Delta_I(c)}$ and $f\big|_{\Delta_I(c')}$. Thus, since R is non-contractible, both are either unbounded or bounded. □

One checks easily that if $I \neq J$, there is no homotopy sending Δ_I to Δ_J; this and Lemma 2.4 motivate the following definition:

Definition 2.5. *Let $f : \mathsf{R}^n \to \mathsf{R}$ be continuous and $I \subset N$. We say that f is I-cofinal (resp. I-bounded) if $f\big|_{\Delta_I}$ is unbounded (resp. bounded).*

Definition 2.6. *The cofinality class of $f : \mathsf{R}^n \to \mathsf{R}$ is the set $\mathfrak{C}(f) \overset{\text{def}}{=} \{I \subset N \ : \ f \text{ is } I\text{-cofinal}\}$.*

Our main results are :

Theorem 1. *Two continuous maps $f, g : \mathsf{R}^n \to \mathsf{R}$ are homotopic if and only if $\mathfrak{C}(f) = \mathfrak{C}(g)$.*

We recall that an antichain in a partially ordered set is a set of pairwise incomparable elements. As usual, we order $\mathcal{P}(N)$ by the inclusion.

Theorem 2. *The homotopy classes $[\mathsf{R}^n, \mathsf{R}]$ of continuous maps $\mathsf{R}^n \to \mathsf{R}$ are in bijection with the antichains of $\mathcal{P}(N) \backslash \{\emptyset\}$.*

It is worth noting that the problem of counting the antichains of $\mathcal{P}(N)$ is NP-complete, see [3]. The exact values for $n = 1, \ldots, 7$ as well as some inequalities can be found in [1].

3 Topology in R^n

We first prove a useful property of "big" open sets, that is, those who contain some $\Delta_I(c)$ outside of a compact set. The formulation given below is slightly more general.

Lemma 3.1 (structure of open sets). *Let $h : \mathsf{R} \to \mathsf{R}^n$ be continuous such that $\pi_i \circ h$ is cofinal for $i \in I$ and bounded otherwise, and let U be an open set in R^n containing $\mathrm{Im} h$. Then, there are $x \in \mathsf{R}$ and $y_j, y'_j \in \mathsf{R}$ ($j \in N \backslash I$) with $y_i < y'_i$, such that*

$$U \supset \prod_{i=1}^{n} U_i,$$

where $U_i = [x, \omega_1[$ if $i \in I$ and $U_j =]\, y_j, y'_j[$ if $j \notin I$.

Notice that x does not depend on i. Given $z \in \mathsf{R}$, it is easy to find h such that $\mathrm{Im} h = \Delta_I(c) \backslash\,]0, z[^n$.

Proof. We may assume that $I = \{1, \ldots, s\}$. Since $\pi_i \circ h$ is bounded for $i = s+1, \ldots, n$, there are $z, c_{s+1}, \ldots, c_n \in \mathsf{R}$ such that $\pi_i \circ h\big|_{[z,\omega_1[} \equiv c_i$. Write $c = (c_{s+1}, \ldots, c_n)$. Suppose (ab absurdo) that for all $x, y_i, y'_i \in \mathsf{R}$ with $y_i < y'_i$,

$$[x, \omega_1[^s \times \left(\prod_{k=s+1}^{n}]y_i, y'_i[\right) \not\subset U.$$

We shall show that this implies that $\Delta_{\{1,\ldots,s\}}(c) \cap (\mathsf{R}^n \backslash U)$ is (closed and) unbounded (in $\Delta_{\{1,\ldots,s\}}(c)$), which is a contradiction since $\mathrm{Im} h \cap \Delta_{\{1,\ldots,s\}}(c)$ is (closed and) unbounded as well. (It is well known that if $g : \mathsf{R} \to \mathsf{R}$ is continuous and cofinal, the set of its fixed points $\{x : g(x) = x\}$ is closed unbounded. Apply Lemma 2.2 to the s first coordinates of h.)
So, let $u \in \mathsf{R}$. For $i = s+1, \ldots, n$, let us fix sequences $y_{i,m} \nearrow c_i$ and $y_{i,m} \searrow c_i$ ($m \in \omega$), and set $x_0 = (u, \ldots, u) \in \mathsf{R}^s$. We choose by induction the sequences $x_m \in \mathsf{R}^s$ and $z_m = (z_{s+1,m}, \ldots, z_{n,m}) \in \mathsf{R}^{n-s}$ such that:

$$x_m \in [\, |x_{m-1}|, \omega_1 [^s, \quad z_{i,m} \in]y_{i,m}, y'_{i,m}[, \quad \text{and } (x_m, z_m) \notin U$$

Then, $z_{i,m} \to c_i$, and $x_m \to (x, \ldots, x)$ for some $x \geq u$. By closeness, $(x, \ldots, x, c) \in \Delta_{\{1,\ldots,s\}}(c) \cap \mathsf{R}^n \backslash U$. ∎

We now prove a kind of analog of Lemma 3.1 for closed sets which will be useful in the proof of Lemma 5.6.

Lemma 3.2 (structure of closed unbounded sets). *Let $F \subset \mathsf{R}^n$ be closed and unbounded. Then, there are $I \subset N$ and $c \in \mathsf{R}^{n-|I|}$ such that $\Delta_I(c) \cap F$ is closed and unbounded (in $\Delta_I(c)$).*

Proof. Let $U = \mathsf{R}^n \backslash F$ and $J = \{i \in N : \pi_i(F) \text{ is unbounded}\}$. We may assume $J = \{1, \dots, s\}$. For $i = s+1, \dots, n$, $\pi_i(F)$ is bounded by $b \in \mathsf{R}$, say. Thus, $F \subset \mathsf{R}^s \times [0, b]^{n-s}$. We show the result by induction on $s = |J|$. Suppose that for all $c = (c_{s+1}, \dots, c_n) \in [0, b]^{n-s}$, $\Delta_J(c) \cap F$ is bounded. Thus, for each such c there is $x(c)$ such that $U \supset (\Delta_J \backslash [0, x(c)[^n)$. By Lemma 3.1, there are $x'(c)$ and $y(c_i) < c_i < y'(c_i)$ such that

$$[x'(c), \omega_1[\times \underbrace{\Big(\prod_{i=s+1}^{n}]y(c_i), y'(c_i)[\Big)}_{V(c)} \subset U.$$

Since $\{V(c)\}_{c \in [0,b]^{n-1}}$ is an open cover of $[0, b]^{n-1}$, there is a finite cover $\{V(c^1), \dots, V(c^m)\}$. Putting $x = \max_{j=1,\dots,m} x'(c^j)$, we get $U \supset [x, \omega_1[^s \times [0, b]^{n-s}$. Letting $Q_i = \{(x_1, \dots, x_s) \in \mathsf{R}^s : x_i \in [0, z]\} \times [0, b]^{n-s}$, we proved that $F \subset \cup_{i=1}^s Q_i$ and then for some $i \in \{1, \dots, s\}$, $F \cap Q_i$ is unbounded. But the number of unbounded projections of $F \cap Q_i$ is at most $s - 1$ (which is a contradiction if $s = 1$ since F is unbounded), and we finish by induction. □

4 Cofinality classes

Lemma 4.1. *Let $I \subset N$ be non-empty. If f is I-cofinal and $J \supset I$, then f is J-cofinal.*

Proof. We may assume that $I = \{1, \dots, s\}$ and $J = \{1, \dots, s'\}$ with $s' \geq s$. Let $z \in \mathsf{R}$, we shall find $x \in \Delta_J$ satisfying $f(x) \geq z$. Put $x_0 = 0$ and define $x_m \in \mathsf{R}$ $(m \in \omega)$ as follows. Given x_{m-1}, take $x_m \geq x_{m-1}$ such that

$$f(\underbrace{x_m, \dots, x_m}_{s}, \underbrace{x_{m-1}, \dots, x_{m-1}}_{s'-s}, \underbrace{0, \dots, 0}_{n-s'}) \geq z$$

(x_m exists since by Lemma 2.4, $f\big|_{\Delta_I(x_{m-1}, \dots, x_{m-1}, 0, \dots, 0)}$ is unbounded). This sequence converge to some x and we have

$$f(\underbrace{x, \dots, x}_{s'}, \underbrace{0, \dots, 0}_{n-s'}) \geq z.$$

□

Lemma 4.2. *If f unbounded then f is N-cofinal.*

Proof. By induction on n. If $n = 1$, the result is trivial. So, let $n > 1$ and suppose that $f\big|_{\Delta_N}$ is bounded, say by $b \in \mathsf{R}$. Fix $b' > b$ and let F be the closed unbounded set $f^{-1}([b', \omega_1[)$. Then, the open set $U = \mathsf{R}^n \backslash F$ contains Δ_N. By Lemma 3.1, there is some $z \in \mathsf{R}$ such that $U \supset [z, \omega_1[^n$. Thus, $F \subset \cup_{i=1}^n Q_i$, where $Q_i = \{(x_1, \ldots, x_n) \in \mathsf{R}^n : x_i \in [0, z]\}$, and $f\big|_{Q_i}$ is unbounded for some i. We fix this i. For $c \in \mathsf{R}$, Let $P_i(c)$ be $\{(x_1, \ldots, x_n) \in \mathsf{R}^n : x_i = c\}$. Then, for all $c \in [0, z]$, $f\big|_{P_i(c)}$ is bounded, otherwise, since $P_i(c)$ is homeomorphic to R^{n-1}, by induction $f\big|_{\Delta_{N\backslash\{i\}}(c)}$ is unbounded, and then by Lemmas 2.4 and 4.1 f is N-cofinal. Let $d(c)$ be a bound for $f\big|_{P_i(c)}$, $\{c_m\}_{m\in\omega}$ be a dense subset of $[0, z]$ and let $d = \sup_{m\in\omega} d(c_m)$. By density and continuity, $f\big|_{P_i(c)}$ is bounded by d for all $c \in [0, z]$, and therefore $f\big|_{Q_i}$ is also bounded by d, contradiction. $\square$

Corollary 4.3. *The cofinality classes of continuous functions* $\mathsf{R}^n \to \mathsf{R}$ *are in bijection with the antichains of* $\mathcal{P}(N)\backslash\{\emptyset\}$.

Proof. By Lemmas 4.1–4.2, the cofinality classes of continuous functions $\mathsf{R}^n \to \mathsf{R}$ are in bijection with the subsets $\mathfrak{I}$ of $\mathcal{P}(N)\backslash\{\emptyset\}$ satisfying the condition that if $I \in \mathfrak{I}$ is non-empty and $J \supset I$, then $J \in \mathfrak{I}$. (Given such a $\mathfrak{I}$ it is easy to find f such that $\mathfrak{C}(f) = \mathfrak{I}$, see below.) To any such $\mathfrak{I}$ corresponds bijectively an antichain given by its minimal elements. The empty antichain corresponds to bounded maps. $\square$

Theorem 2 follows immediately from this corollary and Theorem 1. It is easy to find a representant for each class of maps $\mathsf{R}^n \to \mathsf{R}$:

Definition 4.4. *Let* $\mathfrak{I} = \{I_1, \ldots, I_k\}$ *be an antichain in* $\mathcal{P}(N)\backslash\{\emptyset\}$. *The canonical representant of the cofinality class* $\mathfrak{I}$ *is given by*

$$f_{\mathfrak{I}}(x_1, \ldots, x_n) \stackrel{\text{def}}{=} \begin{cases} \max\limits_{\ell=1,\ldots,k} \left\{ \min\limits_{i\in I_\ell}\{x_i\} \right\} & \textit{if } \mathfrak{I} \neq \emptyset \\ \\ 0 \ \textit{if } \mathfrak{I} = \emptyset. \end{cases} \tag{2}$$

One checks easily that if $\mathfrak{I}$ is an antichain, $\mathfrak{I}$ contains exactly the minimal elements of $\mathfrak{C}(f_{\mathfrak{I}})$. Assuming Theorems 1-2, it is easy to show that $[\mathsf{R}^n, \mathsf{R}^n]$ is isomorphic to a monoid of $(2^n - 1) \times (2^n - 1)$ matrices with entries $0, 1$.

Definition 4.5. *Let* $f : \mathsf{R}^n \to \mathsf{R}^n$ *be continuous. We define its direction matrix* $D(f) = (D_{I,J}(f))_{I,J\in\mathcal{P}(N)\backslash\{0\}}$ *by* $D_{I,J}(f) = 1$ *if there is some* $c \in \mathsf{R}^{n-|J|}$ *such that* $f(\Delta_I) \cap \Delta_J(c)$ *is unbounded in* $\Delta_J(c)$, *and* $D_{I,J}(f) = 0$ *otherwise.*

If $D(f) = D(g)$, then $\mathfrak{C}(\pi_i \circ f) = \mathfrak{C}(\pi_i \circ g)$ for each $i \in N$, thus by Theorem 1, f and g are homotopic, the converse being obviously true. Notice that by continuity, for a fixed I there is at most one J such that $D_{I,J}(f) = 1$.

Proposition 4.6. *If $f, g : \mathsf{R}^n \to \mathsf{R}^n$ are continuous, $D(f \circ g) = D(g) \cdot D(f)$.*

Proof. By taking canonical representants for each coordinate of f, g, (Definition 4.4), we may assume that $f(\Delta_I) \cap \Delta_J$ is unbounded if and only if $D_{I,J}(f) = 1$ (that is, the 'c' of Definition 4.5 is 0), and similarly for g. The proof is then a routine check. □

Of course, not every $(2^n - 1) \times (2^n - 1)$ matrix of $0, 1$ is a direction matrix, there are some restrictions (which seem however quite tedious to describe).

5 Partition properties

The goal of this section is to prove an analog of Lemma 2.2 in [2] which says that if $f : \mathsf{R} \to \mathsf{R}$ is continuous and cofinal, there is a partition $\{P(\beta)\}_{\beta \in \omega_1}$ of R, with $P(\beta) = [x_\beta, x_{\beta+1}]$, such that $f(P(\beta)) = P(\beta)$ for each β. The homotopy question is then reduced to the trivial problem of finding homotopies (here, between f and the identity map) defined in $P(\beta)$ and leaving $\partial P(\beta) = \{x_\beta, x_{\beta+1}\}$ fixed. If $f : \mathsf{R} \to \mathsf{R}$ is bounded, then by Lemma 2.1, f is constant outside a bounded set, and the homotopy question is again trivial.

For maps $f : \mathsf{R}^n \to \mathsf{R}$ with $\mathfrak{C}(f) = \mathcal{P}(N) \backslash \{\emptyset\}$, [2, Lemma 2.2] may be applied to define homotopies in $P(\beta) = [x_\beta, x_{\beta+1}]^n \backslash [0, x_\beta[^n$. Since $\cup_{\alpha \in \omega_1} P(\beta) = \mathsf{R}^n$, it suffices to glue together the homotopies to obtain Theorem 1 in this special case. If f is bounded then f is trivially homotopic to the constant map 0. The problem is more difficult if f is cofinal in some but not all $I \subset N$, but the idea is always to find a partition $\{[x_\beta, x_{\beta+1}]\}_{\beta \in \omega_1}$ of R, such that if $x \in \mathsf{R}^n$ is "between x_β and $x_{\beta+1}$ in a cofinal direction I" (see Definition 5.5 and (8) below), then $f(x) \in [x_\beta, x_{\beta+1}]$. Moreover, f will be constant "along bounded directions" for x sufficiently large (in these directions), see Lemmas 5.1–5.4.

Lemma 5.1. *Let $f : \mathsf{R}^n \to \mathsf{R}$ be bounded. Then, there are $x, c \in \mathsf{R}$ such that $f\big|_{[x,\omega_1[^n} \equiv c$.*

Proof. For $n = 1$, this is Lemma 2.1. If $n \geq 1$, since $f\big|_{\Delta_N}$ is bounded, there are $(x, \ldots, x) \in \Delta_N$ and $c \in \mathsf{R}$ such that for all $x' \geq x$,

$f(x', \ldots, x') = c$. Let $c_m < c < c'_m$ be sequences converging to c. By Lemma 3.1, since $f^{-1}(]c_m, c'_m[) \supset (\Delta_N \setminus [0, x[^n)$, there is x_m such that $f^{-1}(]c_m, c'_m[) \supset [x_m, \omega_1[^n$. Thus, $f^{-1}(c) = \bigcap_m f^{-1}(]c_m, c'_m[) \supset [\sup_m x_m, \omega_1[^n$. □

It is useful to introduce the following notation:

Definition 5.2. *If $I \subset N$, $c \in \mathsf{R}$, we set $M_I(c) \stackrel{\text{def}}{=} \{x \in \mathsf{R}^n \,:\, x_I \in [c, \omega_1[^{|I|}\}$.*

Lemma 5.3. *Let $I \subset N$ be non-empty.*

1) *Suppose that f is I-bounded. Then, for all $c \in \mathsf{R}$, there is $d(c)$ minimal such that for all $b \in [0, c]^{n-|I|}$, f restricted to*

$$E_I(b, d(c)) \stackrel{\text{def}}{=} \{x \in M_I(d(c)) \,:\, x_{N \setminus I} = b\} \tag{3}$$

is constant.

2) *Suppose that f is I-cofinal. Then, for all $c \in \mathsf{R}$, there is $\widetilde{d}(c) \in \mathsf{R}$ minimal such that for all $b \in [0, c]^{n-|I|}$, $f(\Delta_I(b) \cap M_I(\widetilde{d}(c))) \subset [c, \omega_1[$.*

Proof. 1) By Lemmas 2.4 and 5.1, for all $b \in [0, c]^{n-|I|}$ there is $d'(b)$ such that f restricted to $E_I(b, d'(b))$ is constant. Let $\{b_m\}_{m \in \omega}$ be a dense subset of $[0, c]^{n-|I|}$. Then $d(c) = \sup_{m \in \omega} d'(b_m)$ has the required property. If $d(c)$ is not minimal, take the minimal one (which exists by continuity of f).

2) Let $b \in [0, c]^{n-|I|}$; since $\Delta_I(b)$ is homeomorphic to R, by [2, Lemma 2.2] there is $\widetilde{d'}(b)$ such that $f(\Delta_I \cap M_I(\widetilde{d'}(b))) \subset [c, \omega_1[$. As in 1), $\widetilde{d}(c) = \sup_{m \in \omega}(\widetilde{d'}(b_m))$ has the required property if $\{b_m\}_{m \in \omega}$ is dense in $[0, c]^{n-|I|}$. □

Lemma 5.4. *Assume that f is I-bounded. In the notations of Lemma 5.3, let $p_I^{\mathrm{bd}}(c) = \max\{c, d(c)\}$. Then, $p_I^{\mathrm{bd}}\big|_{\omega_1}$ is monotone increasing and continuous, and $\{c \in \mathsf{R} \,:\, p_I^{\mathrm{bd}}(c) = c\}$ contains a closed unbounded set. If f is I-cofinal, then $p_I^{\mathrm{cf}}(c) = \max\{c, \widetilde{d}(c)\}$ has the same properties.*

It is not true in general that p_I^{cf} and p_I^{bd} are continuous in R. We will use the fact that in ω_1 we only have limits "from below".

Proof. Assume that f is I-bounded. We first prove that p_I^{bd} is continuous (monotonicity is clear by definition). It is enough to prove it for $d(c)$. Let $\alpha_m \in \omega_1 \subset \mathsf{R}$ ($m \in \omega$) be a sequence converging to $\alpha \in \omega_1$. We may assume that for each m, $\alpha_m \leq \alpha$. Let $d'(\alpha) = \lim_{m \to \infty} d(\alpha_m)$.

By monotonicity of $d(\cdot)$, the limit exists and $d'(\alpha) \leq d(\alpha)$. By minimality of $d(\alpha)$, it is enough to show that for each $b \in [0,\alpha]^{n-|I|}$, f restricted to $E_I(b, d'(\alpha))$ is constant.
Assume for simplicity that $I = \{1,\dots,s\}$. Let $x = (x_1,\dots,x_n) \in E_I(b,d'(\alpha))$, that is, $x_1,\dots,x_s \geq d'(\alpha)$ and $(x_{s+1},\dots,x_n) = b$. For each $m \in \omega$, choose $b_m \in [0,\alpha_m]^{n-s}$ such that $b_m \to b$. Put $x_m = (x_1,\dots,x_s,b_m)$. Since $d'(\alpha) \geq d(\alpha_m)$, $x_m \in E_I(b_m, d(\alpha_m))$, so $f(x_m) = f(y_m)$ where $y_m = (d(\alpha_m),\dots,d(\alpha_m),b_m)$. By continuity, $f(x) = f(d'(\alpha),\dots,d'(\alpha),b)$. Since this holds for all x in $E_I(b,d'(\alpha))$, f is constant on this set.
We now prove that $K = \{\alpha \in \omega_1 \subset \mathsf{R} : p_I^{\mathrm{bd}}(\alpha) = \alpha\}$ is closed unbounded. Closeness is immediate by continuity. Let $\alpha_0 \in \omega_1$. Define inductively $\alpha_m \in \omega_1$ ($m \in \omega$) such that $\alpha_m \geq p_I^{\mathrm{bd}}(\alpha_{m-1})$. By continuity, $\lim_{m\to\infty} \alpha_m = \alpha = p_I^{\mathrm{bd}}(\alpha) \geq \alpha_0$. This shows that K is unbounded. The proof for p_I^{cf} is similar. $\square$

We have now taken care of the bounded directions. We proceed with the investigation of cofinal directions. Lemma 5.4 for p_I^{cf} will be helpful.

Definition 5.5. *Let $I \subset N$ and $\alpha \in \omega_1$. We set:*

$$A_I^-(\alpha) \overset{\mathrm{def}}{=} \left\{ x \in \mathsf{R}^n \ : \ \begin{array}{l} x_I \in [0,\alpha]^{|I|},\ \textit{and} \\ x_j \leq |x_I|\, \forall j \in N \end{array} \right\},$$
$$A_I^+(\alpha) \overset{\mathrm{def}}{=} \left\{ x \in \mathsf{R}^n \ : \ \begin{array}{l} x_I \in [\alpha,\omega_1[^{|I|},\ \textit{and} \\ x_j \leq |x_I|\, \forall j \in N \end{array} \right\}.$$

The following lemma is the key argument for proving Theorem 1.

Lemma 5.6. *If f is I-cofinal, then*

$$\{\alpha \in \omega_1 \ : \ f(A_I^+(\alpha)) \subset [\alpha,\omega_1[\ \textit{and}\ f(A_I^-(\alpha)) \subset [0,\alpha]\}$$

is closed and unbounded.

Proof. 1) We first show that $F^- = \{\alpha \in \omega_1 \ : \ f(A_I^-(\alpha)) \subset [0,\alpha]\}$ is closed and unbounded. Let $\alpha_m \to \alpha$, $m \in \omega$. One may assume $\alpha_m \leq \alpha$. Since $\overline{\cup_{m\in\omega} A_I^-(\alpha_m)} = A_I^-(\alpha)$, by continuity $f(A_I^-(\alpha)) \subset \overline{(f(\cup_{m\in\omega} A_I^-(\alpha_m))} \subset [0,\alpha]$. Thus, F^- is closed. We now show that $\{\alpha \ : \ f([0,\alpha]^n) \subset [0,\alpha]\}$ is unbounded, which implies that F^- is unbounded since $A_I^-(\alpha) \subset [0,\alpha]^n$. So, let $\beta_0 \in \omega_1$. For $m \in \omega$, we define $\beta_m \geq \beta_{m-1}$ such that $f([0,\beta_{m-1}]^n) \subset [0,\beta_m]$ (f being continuous, $f([0,\beta_{m-1}]^n)$ is compact and thus bounded). Then, $\lim_{m\to\infty} \beta_m = \beta \in F^-$.

2) We now show that $F^+ = \{\alpha \in \omega_1 \ : \ f(A_I^+(\alpha)) \subset [\alpha,\omega_1[\}$ is closed and unbounded. The proof that F^+ is closed is like in 1), using

$\cap_{m\in\omega} A_I^+(\alpha_m) = A_I^+(\alpha)$. To prove that F^+ is unbounded, we use the following claim:

Claim. *For all $\alpha \in \omega_1$, there is $\beta(\alpha) \geq \alpha$ such that $f(A_I^+(\beta(\alpha))) \subset [\alpha, \omega_1[$.*

This suffices to finish the proof: given α_0, we define the sequence $\alpha_m = \beta(\alpha_{m-1})$ $(m \in \omega)$ whose limit $\alpha \geq \alpha_0$ is in F^+.

Proof of the claim. To simplify, assume that $I = \{1, \ldots, s\}$. Ab absurdo, suppose that:

$$\forall \alpha, \forall \beta \geq \alpha,\ \exists x_\beta \in A_I^+(\beta) \text{ with } f(x_\beta) \leq \alpha. \tag{4}$$

For each β, we fix $x_\beta \in A_I^+(\beta)$ such that (4) holds. We now proceed in several steps.
a) We first show that

$$\Gamma = \left\{ \gamma : \begin{array}{l} \exists b_{s+1}, \ldots, b_n \in \mathsf{R} \text{ such that} \\ f(\gamma, \ldots, \gamma, b_{s+1}, \ldots, b_n) \in [0, \alpha] \end{array} \right\}$$

is closed and unbounded. Indeed, given $\gamma_m \to \gamma$, $\gamma_m \in \Gamma$, there corresponds sequences $b_{j,m}$ $j = s+1, \ldots, n$. Taking convergent subsequences, we see that $\gamma \in \Gamma$, which is thus closed. Now, given $\beta \in \omega_1$, we may define $\gamma_0 = \max\{\alpha, \beta\}$. Then, by induction, choose $\gamma_m \geq |x_{\gamma_{m-1}}|$; thus $f(x_{\gamma_m}) \leq \alpha$ (recall that each x_β satisfies (4)). Taking a convergent subsequence of the x_{γ_m}, we obtain an $x = (\gamma, \ldots, \gamma, b_{s+1}, \ldots, b_n)$ with $\gamma \geq \beta$ and $f(x) \leq \alpha$, showing that Γ is unbounded.

b) We then set:

$$C(\gamma) \stackrel{\text{def}}{=} \left\{ b = (b_{s+1}, \ldots, b_n) \in \mathsf{R}^{n-s} : \begin{array}{l} \exists \beta \geq \gamma \text{ with} \\ f(\beta, \ldots, \beta, b) \leq \alpha \end{array} \right\}.$$

$C(\gamma)$ is nonempty, closed, and if $\gamma' \geq \gamma$, $C(\gamma') \subset C(\gamma)$.

c) For all $\gamma \in \omega_1$, $C(\gamma)$ is unbounded. Indeed, Lemma 5.4 (for p_I^{cf}) implies that the set of β satisfying $f(\Delta_I(b) \cap M_I(\beta)) \subset [\beta, \omega_1[$ for all $b \in [0, \beta]^{n-s}$ is unbounded. Thus, if $C(\gamma)$ is bounded, $C(\gamma) \subset [0, \beta]^{n-s}$ for one such $\beta > \alpha$, and then $C(\beta) \subset [0, \beta]^{n-s}$, which implies that $C(\beta)$ is empty, contradicting b).

d) By c) and Lemma 3.2, for all $\gamma \in \omega_1$ the set

$$\mathfrak{J}(\gamma) = \left\{ J \subset \{s+1, \ldots, N\} : \begin{array}{l} \exists c \in \mathsf{R}^{n-s-|J|} \text{ such that} \\ \Delta_J(c) \cap C(\gamma) \text{ is unbounded} \end{array} \right\}$$

is nonempty. Since $\mathfrak{J}(\gamma') \subset \mathfrak{J}(\gamma)$ if $\gamma' > \gamma$, $\cap_{\gamma\in\omega_1} \mathfrak{J}(\gamma)$ is also nonempty, let J be in this intersection. We may assume $J = \{s+1, \ldots, s_1\}$. So,

for all $\gamma \in \omega_1$, and all $x \in \mathsf{R}$, there is $y \geq x$ and $c_{s_1+1}, \ldots, c_n$ such that $(y, \ldots, y, c_{s_1+1}, \ldots, c_n) \in C(\gamma)$. In other words, for any γ, there are $y \geq \gamma$, $\beta \geq \gamma$ and $c_{s_1+1}, \ldots, c_n$ such that

$$f(\underbrace{\beta, \ldots, \beta}_{s}, \underbrace{y, \ldots, y}_{s_1-s}, c_{s_1+1}, \ldots, c_n) \leq \alpha. \tag{5}$$

Given γ_0, we may define sequences $\gamma_m, \beta_m, y_m, c_{s_1+1,m}, \ldots, c_{n,m}$ by letting $\gamma_m \geq \max\{\gamma_{m-1}, y_{m-1}\}$ and choosing $y_m \geq \gamma_m$, $\beta_m \geq \gamma_m$ and $c_{s_1+1,m}, \ldots, c_{n,m}$ satisfying (5). We thus have $\gamma_m \geq \beta_{m-1} \geq \gamma_{m-1}$ and $\gamma_m \geq y_{m-1} \geq \gamma_{m-1}$, these three sequences converge to the same point γ. Taking convergent subsequences of $c_{s_1+1,m}, \ldots, c_{n,m}$, we have found an $x = (\gamma, \ldots, \gamma, c_{s_1+1}, \ldots, c_n) \in \mathsf{R}^n$ such that $\gamma \geq \gamma_0$ and $f(x) \leq \alpha$. That is, we proved that

$$\Gamma_1 = \left\{ \gamma : \begin{array}{l} \exists b_{s_1+1}, \ldots, b_n \in \mathsf{R} \text{ such that} \\ f(\gamma, \ldots, \gamma, b_{s_1+1}, \ldots, b_n) \in [0, \alpha] \end{array} \right\}$$

is unbounded, and its closeness is immediate.

e) We may thus go back to b) with s_1 instead of s, and proceed by induction until we obtain $\{\gamma : f(\gamma, \ldots, \gamma) \leq \alpha\}$ is closed and unbounded. (In c), we use Lemma 4.1 which ensures that f is $\{1, \ldots, s_1\}$-cofinal.) But by Lemma 5.4 for p_N^{cf}, this implies that $f\big|_{\Delta_N}$ is bounded, which is contradicts Lemma 4.1 since f is I-cofinal. This proves the claim.

∎

6 Proof of the main theorem

We will now use our partition properties to define an homotopy between f and the canonical representant of its cofinality class. We first recall the following triviality:

Lemma 6.1. *Let $g, h : X \to Y$ be continuous and Y be homeomorphic to $[0,1]^d$. Then, there is a homotopy ϕ_t such that $\phi_0 = g, \phi_1 = h$ and for all t, $\phi_t\big|_Q = id$, where $Q = \{x \in X : f(x) = g(x)\}$.*

Proof. Let $\varphi : Y \to [0,1]^d$ be an homeomorphism. Then, $\phi_t(x) = \varphi^{-1}\Big(\varphi(g(x)) \cdot (1-t) + \varphi(h(x)) \cdot t\Big)$ has the required properties. ∎

Proof of Theorem 1. Let $f : \mathsf{R}^n \to \mathsf{R}$ be continuous. The case $\mathfrak{C}(f) = \emptyset$ (and thus f bounded) being trivial, we may assume $\mathfrak{C}(f) \neq \emptyset$. To clarify the exposition, we fix some map $h : \mathsf{R}^2 \to \mathsf{R}$ with $\mathfrak{C}(h) = \{\{1\}, \{1,2\}\}$ to serve us as an example. We shall carry the proof

for general $f : \mathsf{R}^n \to \mathsf{R}$ and for h together. Let $\mathfrak{J}$ be the minimal elements of $\mathfrak{C}(f)$. We shall show that f and $f_{\mathfrak{J}}$ (defined by (2)) are homotopic. In the case of h, $\mathfrak{J} = \{\{1\}\}$ and $f_{\mathfrak{J}}$ is the projection on the first coordinate.
By Lemmas 2.2, 5.4 and 5.6, the set Θ of ordinals α satifying

$$\forall I \in \mathfrak{C}(f), \quad f(A_I^+(\alpha)) \subset [\alpha, \omega_1[,\ f(A_I^-(\alpha)) \subset [0, \alpha], \tag{6}$$

and

$$\forall J \notin \mathfrak{C}(f), \quad \alpha = p_J^{\mathrm{bd}}(\alpha) \tag{7}$$

is closed and unbounded. For all $\beta \in \omega_1$, we then choose $\alpha_\beta \in \Theta$ as follows:

$$\begin{aligned} \alpha_{\beta+1} &= \min \Theta \cap [\alpha_\beta + 1, \omega_1[, \\ \alpha_\beta &= \sup_{\gamma<\beta} \alpha_\gamma \text{ if } \beta \text{ is a limit ordinal.} \end{aligned}$$

Then, for all $I \in \mathfrak{C}(f)$, we set $P_I(\beta) = A_I^+(\alpha_\beta) \cap A_I^-(\alpha_{\beta+1})$, that is, $P_I(\beta) = \{x \in \mathsf{R}^n : x_I \in [\alpha_\beta, \alpha_{\beta+1}]^{|I|} \text{ and } x_j \leq |x_I|\, \forall j \in N\}$. Finally, set

$$P(\beta) = \bigcup_{I \in \mathfrak{C}(f)} P_I(\beta). \tag{8}$$

The corresponding sets for h are shown on Figure 1. By (6), for

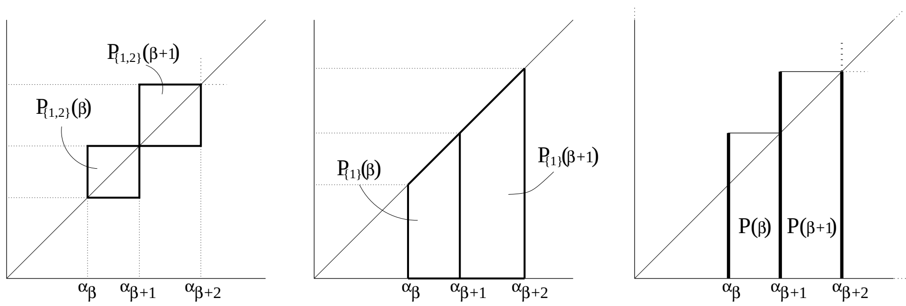

Figure 1: The sets $P_I(\alpha)$ ($I = \{1\}, \{1,2\}$; $\alpha = \beta, \beta+1$) for h.

$I \in \mathfrak{C}(f)$ and all β, $f(P_I(\beta)) = [\alpha_\beta, \alpha_{\beta+1}]$. Notice that we also have $f_{\mathfrak{J}}(P_I(\beta)) = [\alpha_\beta, \alpha_{\beta+1}]$. One easily checks that $P_I(\beta) \cap P_I(\beta+1) = \{x \in \mathsf{R}^n : x_i = \alpha_{\beta+1} \text{ for } i \in I \text{ and } x_j \leq \alpha_{\beta+1}\, \forall j \in N\}$, and thus $f(P_I(\beta) \cap P_I(\beta+1)) = \{\alpha_{\beta+1}\}$; if $I, J \in \mathfrak{C}(f)$, then $P_I(\beta) \cap P_J(\beta) \subset P_{I \cup J}(\beta)$ (recall that by Lemma 4.1 $I \cup J \in \mathfrak{C}(f)$), and $P_I(\beta) \cap P_J(\beta+$

$1) \subset P_J(\beta) \cap P_J(\beta+1)$. Thus,

$$\begin{aligned} P(\beta) \cap P(\beta+1) &= \bigcup_{I \in \mathfrak{C}(f)} (P_I(\beta) \cap P_I(\beta+1)) \\ &= \bigcup_{I \in \mathfrak{C}(f)} \left\{ x \in \mathsf{R}^n \ : \ \begin{array}{c} x_i = \alpha_{\beta+1} \text{ for } i \in I, \\ x_j \leq \alpha_{\beta+1} \, \forall j \in N \end{array} \right\}, \end{aligned}$$

and

$$f(P(\beta) \cap P(\beta+1)) = \{\alpha_{\beta+1}\} = f_{\mathfrak{J}}(P(\beta) \cap P(\beta+1)). \tag{9}$$

For h, this means that the bold vertical boundaries of the rightmost picture in Figure 1 are "projected down".
We can now apply Lemma 6.1 with $X = P(\beta)$ and $Y = [\alpha_\beta, \alpha_{\beta+1}]$ to find homotopies ϕ_t^β defined on $P(\beta)$ such that $\phi_0^\beta = f\big|_{P(\beta)}$ and $\phi_1^\beta = f_{\mathfrak{J}}\big|_{P(\beta)}$. By (9), if $x \in P(\beta) \cap P(\beta+1)$, $\phi_t^\beta(x) = \phi_t^{\beta+1}(x) = \alpha_{\beta+1}$ for all $t \in [0,1]$, and we can "glue" together the ϕ_t^β to obtain an homotopy ϕ_t between f and $f_{\mathfrak{J}}$ defined on $P = \bigcup_{\beta \in \omega_1} P(\beta) \subset \mathsf{R}^n$. We shall now explain how to extend this homotopy to all of R^n.
First, consider our example h. We have depicted the situation of $P(\beta)$ in Figure 2. By Lemma (7), h restricted to any vertical line depicted in Figure 2 is constant (these vertical lines are exactly the $E_{\{2\}}(b, \alpha_{\beta+1})$ for $b \in [\alpha_\beta, \alpha_{\beta+1}]$). We can then define $R(x) \in \partial P$ for $x \notin P$ as in this figure, and then $h(x) = h(R(x))$. Since the vertical boundaries of $P(\beta)$ are both mapped by h to one point, the ambiguity of the definition of $R(x)$ for x lying on one of the dashed lines of Figure 2 does not cause any trouble. If we extend R by the identity in P, $R(x)$ will be non-continuous (due to the above ambiguities), but $\widetilde{\phi}_t(x) = \phi_t((R(x))$ is continuous, and is then an homotopy between h and $f_{\{1\}}$ (since $f_{\{1\}}(x) = f_{\{1\}}(R(x))$ as well). One sees easily that for fixed t, $\widetilde{\phi}_t$ it is constant on the verticals depicted and fixes $P(\beta) \cap P(\beta+1)$ for all t. Moreover, limit ordinals β do not cause any trouble. (Strictly speaking, the homotopy is not defined in $[0, \alpha_0] \times \mathsf{R}$, but we may squish this set continuously to $\{\alpha_0\} \times \mathsf{R}$ and no bother about it.)

Let us do the general case, i.e. go back to $f : \mathsf{R}^n \to \mathsf{R}$. The only difference is a heavier formalism and no new idea is needed, we shall thus pass quickly over the details. Let $x \in \mathsf{R}^n$, with coordinates $x_{i_1} \geq x_{i_2} \geq \cdots \geq x_{i_n}$. Choose β such that $x_{i_1} \in [\alpha_\beta, \alpha_{\beta+1}]$. (The fact that β is not always unique is not important.) Let k be maximal such that $x_{i_1}, x_{i_2} \dots, x_{i_k} \in [\alpha_\beta, \alpha_{\beta+1}]$. If $I = \{i_1, \dots, i_k\} \in \mathfrak{C}(f)$, then $x \in P_I(\beta) \subset P$. If $I \notin \mathfrak{C}(f)$, choose $\ell \geq k$ maximal such that $I' = \{i_1, \dots, i_\ell\} \notin \mathfrak{C}(f)$ and for some β', $x_{i_1}, \dots, x_{i_\ell} \geq \alpha_{\beta'}$. Then, by definition, $x \in E_{I'}(b, \alpha_{\beta'})$ where $b = (x_{i_{\ell+1}}, \dots, x_{i_n}) \in [0, \alpha_{\beta'}]^{n-\ell}$

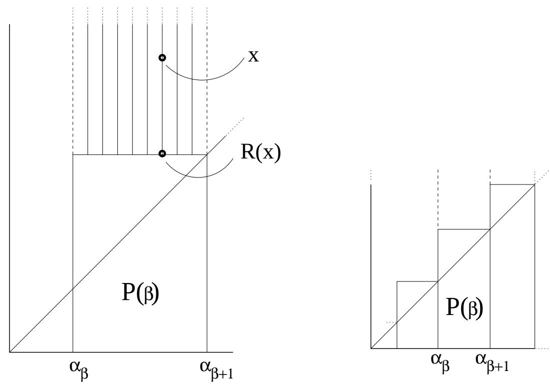

Figure 2: Constant verticals and $R(x)$ for h.

(see (3)). We can choose β' minimal, and then β' is successor if $x_{i_{\ell+1}} < \alpha_{\beta'}$. Then, by minimality of β', $x_{i_{\ell+1}} \in]\alpha_{\beta'-1}, \alpha_{\beta'}[$. Set then $R(x) = (R_1(x), \ldots, R_n(x))$ by letting $R_{i_s}(x) = x_{i_s}$ for $s = 1, \ldots, \ell$ and $R_{i_s}(x) = \alpha_{\beta'-1}$ for $s = \ell+1, \ldots, n$. By maximality of ℓ, $J = \{i_1, \ldots, i_{\ell+1}\} \in \mathfrak{C}(f)$, and $R(x) \in \partial P_J(\beta'-1)$. If $x_{i_{\ell+1}} = \alpha_{\beta'}$, we set $R_{i_s}(x) = \alpha_{\beta'}$ for $s = \ell+1, \ldots, n$, and then $R(x) \in \partial P_J(\beta')$. By (7) and Lemma 5.4 for p_I^{bd}, $f(x) = f(R(x))$ in both cases. Extending $R(x)$ to all R^n by $R(x) = x$ for x in P, we may define $\widetilde{\phi}_t(x) = \phi_t(R(x))$ and check as in h's case that $\widetilde{\phi}_t$ is continuous and sends f to $f_{\mathfrak{J}}$. $\square$

7 Other manifolds

In this section we consider some other non-metrizable manifolds and state some theorems about their homotopy classes.

Recall first that the long line L is the union of two copies $\mathsf{L}^+, \mathsf{L}^-$ of R glued at 0. In order to code maps $\mathsf{L}^n \to \mathsf{R}$, we let $N^\pm$ be the set of "signed" coordinates $\{+1, \ldots, +n, -1, \ldots, -n\}$ and say that $I \subset N^\pm$ is an admissible subset of $N^\pm$, which we denote by $I \subset^a N^\pm$, if for all $i \in N$, I does not contain both $+i$ and $-i$. We then set $\mathcal{P}^a(N^\pm) = \{I \subset^a N^\pm\}$. We have the following result:

Theorem 3. $[\mathsf{L}^n, \mathsf{R}]$ *is in bijection with the antichains of* $\mathcal{P}^a(N^\pm) \backslash \{\emptyset\}$, *and* $[\mathsf{R}^n, \mathsf{L}]$ *is the union of* $[\mathsf{R}^n, \mathsf{L}^+]$ *and* $[\mathsf{R}^n, \mathsf{L}^-]$ *where bounded maps in* L^+ *and* L^- *are identified.*

Proof. The assertion about $[\mathsf{L}^n, \mathsf{R}]$ is proved as Theorem 2. For $[\mathsf{R}^n, \mathsf{L}]$, notice that a continuous map $\mathsf{R} \to \mathsf{L}$ cannot be unbounded in both L^+ and L^-. Thus, if $f\big|_{\Delta_N}$ is cofinal in L^+, $f\big|_{\Delta_I}$ cannot be cofinal in L^- by Lemma 4.1, and the result follows thus from Theorem 2. $\square$

The homotopy classes of maps $\mathsf{L}^n \to \mathsf{L}$ can be classified as well, but are harder to describe.

Let us give a few more examples in dimension 2. Let C be the set $\{(x,y) \in \mathsf{R}^2 : y \leq x\}$. Fix $k \in \omega$, and set $\overline{i} = i \mod (k+1)$ for $i \in \omega$. Given a finite sequence $S = (s_1, \ldots, s_k)$ of symbols $\uparrow$ and $\downarrow$, we define the S-pipe

$$\mathsf{P}_S = \bigcup_{i=1}^{k} C \times \{i\} / \sim \tag{10}$$

where $x \sim y$ iff $x = y$ or

$$\begin{aligned}
&x = ((u,u),i),\ y = ((u,0),\overline{i+1}) \text{ and } s_i = s_{\overline{i+1}} = \uparrow, && \text{or}\\
&x = ((u,u),i),\ y = ((u,u),\overline{i+1}) \text{ and } s_i = \uparrow,\ s_{\overline{i+1}} = \downarrow, && \text{or}\\
&x = ((u,0),i),\ y = ((u,0),\overline{i+1}) \text{ and } s_i = \downarrow,\ s_{\overline{i+1}} = \uparrow, && \text{or}\\
&x = ((u,0),i),\ y = ((u,u),\overline{i+1}) \text{ and } s_i = s_{\overline{i+1}} = \downarrow,
\end{aligned}$$

For instance, $\mathsf{L}^2 = \mathsf{P}_{(\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow)}$. See Figure 3 for other examples. Any

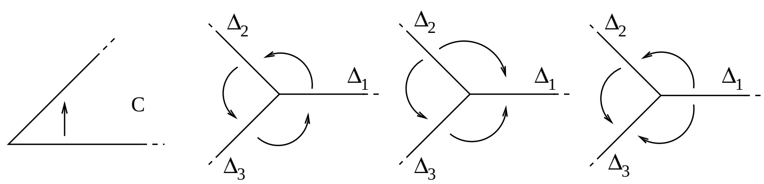

Figure 3: P_S for $S = \{\uparrow\uparrow\uparrow\}, \{\downarrow\uparrow\uparrow\}, \{\uparrow\uparrow\downarrow\}$.

such P_S is a long pipe if we remove one point (see [6, Def 5.2]). It is not difficult to see that P_S and $\mathsf{P}_{S'}$ are homeomorphic iff S and S' differ only by a circular permutation and/or a uniform exchange of $\uparrow$ and $\downarrow$. We may also prove that there are exactly k homotopy classes of unbounded maps $\mathsf{R} \to \mathsf{P}_S$, canonical representants being given by $x \mapsto \pi((x,0),i)$ if $s_i = \uparrow$ and $x \mapsto \pi((x,x),i)$ if $s_i = \downarrow$ (π denotes the projection of $\bigcup_{i=1}^k C \times \{i\}$ on the quotient space P_S). We denote (the image of) these representants by $\Delta_1, \ldots, \Delta_k$ (see Figure 3), and define i-cofinality and i-boundedness of maps $\mathsf{P}_S \to \mathsf{R}$ as in Section 2. S gives a partial order $\prec$ on $\{1, \ldots, k\}$ defined by $i \prec j$ iff $j = \overline{i+\ell}$ for some $\ell \leq k$ and $s_{\overline{i}} = \cdots = s_{\overline{i+\ell-1}} = \uparrow$, or $i = \overline{j+\ell}$ for some $\ell \leq k$ and $s_{\overline{j+1}} = \cdots = s_{\overline{j+\ell}} = \downarrow$ (in other words, if we can pass from i to j following arrows). It is possible that $i \prec i$, if all s_j are equal. As in Lemmas 4.1–4.2, we can prove that if f is i-cofinal and $i \prec j$, then f is j-cofinal, and that an unbounded map is j-cofinal for some (maximal)

j. Applying the technique of Theorems 1–2, we obtain the following result:

Theorem 4. *With the above notations,* $[\mathsf{P}_S, \mathsf{R}]$ *is in bijection with the antichains of the partially ordered set* $\langle\{1,\dots,k\},\prec\rangle$.

For general non-metrizable manifolds, a complete description of its homotopy classes may be quite difficult, even if all $\Pi_i(M)$ are zero, since it may happen that $[M,M]$, or even $[M,\mathsf{R}]$, is infinite (for instance, glue an infinite number of C together in the same fashion as the P_S above). In all generality, what we can say is for instance the following.

Proposition 7.1. *Let M be a manifold such that $\omega_1 \subset M$ (that is, there is an embedding $e : \omega_1 \to M$). Then, M is not contractible.*

It may be interesting to see if we can weaken the hypotheses to "M is non-metrizable", since there are many non-metrizable manifolds that do not contain ω_1. For instance, there are smoothings of R such that the tangent bundle with the 0 section removed does not contain any copy of ω_1 (see [7, class 7, p. 158]). Notice that the assumption that M is a manifold is essential: the cone over ω_1 is contractible.

Proof. We do not make the distinction between $\alpha \in \omega_1$ and $e(\alpha) \in M$, denoting both by α, and identify ω_1 and $e(\omega_1)$. Suppose that there is a continuous $h : [0,1] \times X \to X$ with $h_0 = id$ and $h_1 \equiv y$ for some $y \in M$. Let U be a chart around y. Notice that $\omega_1 \not\subset U$. Then, there is $s < 1$ such that for all $s < t \leq 1$, $h_t(\omega_1) \subset U$. (Otherwise take a sequence $t_m \to 1$, $m \in \omega$. For each m there is $\alpha_m \in \omega_1$ with $h_{t_m}(\alpha_m) \notin U$. By taking a convergent subsequence of the α_m, we obtain $\alpha \in \omega_1$ for which $h_1(\alpha) \notin U$, contradiction.)
Then, for $t > s$, $h_t|_{\omega_1}$ is eventually constant, i.e. $\exists \alpha \in \omega_1$ such that $\forall \beta \geq \alpha$, $h_t(\beta) = h_t(\alpha)$ (U being homeomorphic to some bounded open set in $\mathbb{R}^d$, we may apply a modified version of Lemma 2.1.) We say that $h_t|_{\omega_1}$ is α-eventually constant. Let:

$$\tau \overset{\text{def}}{=} \inf\{t \in [0,1] \ : \ h_t|_{\omega_1} \text{ is eventually constant}\}.$$

We saw that $\tau < 1$. There are two possibilities.

1) There is α such that $h_\tau|_{\omega_1}$ is α-eventually constant. Then, since $h_0 = id$, $\tau > 0$. Choose a sequence $t_m \nearrow \tau$, $m \in \omega$. Since $h_{t_m}|_{\omega_1}$ is non-eventually constant, if V is a chart around $h_\tau(\alpha)$, $\exists \beta_n \geq \alpha$ with $h_{t_n}(\beta_n) \notin V$. (Otherwise, like before, $h_{t_n}|_{\omega_1}$ would be eventually constant.) Taking a convergent subsequence of the β_m, we obtain $\beta \geq \alpha$ with $h_\tau(\beta) \notin V$, contradiction with $h_\tau(\beta) = h_\tau(\alpha)$.

2) $h_\tau|_{\omega_1}$ is not eventually constant. Since $\tau < 1$, let $t_m \searrow \tau$, $m \in \omega$. For each m, there is α_m with $h_{t_m}|_{\omega_1}$ α_m-eventually constant. Taking a subsequence converging to α, we obtain that $h_\tau|_{\omega_1}$ is α-eventually constant, contradiction.

Therefore, such an h_t cannot exist and M is not contractible. □

Acknowledgements. I wish to thank David Cimasoni and Patrick Lapp.

References

[1] I. Anderson. *Combinatorics of finite sets.* The Clarendon Press, Oxford University Press. Oxford Science Publications, New York, 1987.

[2] M. Baillif and D. Cimasoni. Long line knots. To appear in Arch. Math., 2002.

[3] M. Ball and J. Provan. The complexity of counting cuts and of computing the probability that a graph is connected. *Siam J. Comput.*, 12(4):777–788, 1983.

[4] D. Gauld. Homeomorphisms of 1-manifolds and ω-bounded 2-manifolds. *Ann. New York Acad. Sci.*, 704:142–149, 1993.

[5] K. Kunen. *Set theory, an introduction to independance proofs (reprint).* Number 102 in Studies in Logic and the Foundations of Mathematics. North-Holland, Amsterdam, 1983.

[6] P. Nyikos. The theory of nonmetrizable manifolds. In *Handbook of Set-Theoretic Topology*, pages 633–684. North-Holland, Amsterdam, 1984.

[7] P. Nyikos. Various smoothings of the long line and their tangent bundles. *Adv. Math.*, 93(2):129–213, 1992.

[8] G.W. Whitehead. *Elements of homotopy theory.* Springer-Verlag, New York-Berlin, 1978.

Mathieu Baillif
Section de Mathématiques
2-4 rue du Lièvre
1211 Genève 24
Switzerland
`Mathieu.Baillif@math.unige.ch`